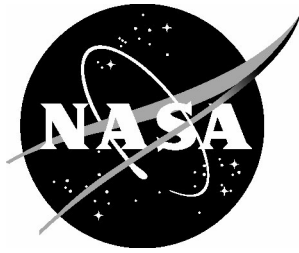


NASA/CR-2004-213002



# A Most Probable Point-Based Method for Reliability Analysis, Sensitivity Analysis and Design Optimization

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February 2004

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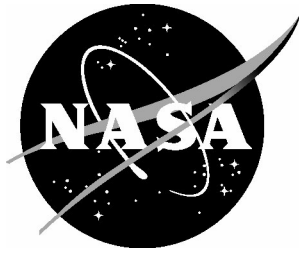
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# A Most Probable Point-Based Method for Reliability Analysis, Sensitivity Analysis and Design Optimization

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Prepared for Langley Research Center  
under Purchase Order L-18606

February 2004

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## ABSTRACT

*A major step in a most probable point (MPP)-based method for reliability analysis is to determine the MPP. This is usually accomplished by using an optimization search algorithm. The minimum distance associated with the MPP provides a measurement of safety probability, which can be obtained by approximate probability integration methods such as FORM or SORM. The reliability sensitivity equations are derived first in this paper, based on the derivatives of the optimal solution. Examples are provided later to demonstrate the use of these derivatives for better reliability analysis and reliability-based design optimization (RBDO).*

## I. INTRODUCTION

Generally, sampling and approximate integration are two commonly used approaches for reliability analysis of a general response or a limit-state equation. The Monte Carlo simulation plays a key role in the sampling approach, whereas the concept of MPP does the same in the approximate integration approach. Derivatives of quantities resulting from reliability analysis can also be computed by either the sampling approach [1] or the approximate integration [2]. The focus of this paper is on the sensitivity analysis pertaining to the approximate integration approach.

As revealed by the Taylor's series expansion, derivatives of a function provide a means to approximate the function beyond the current design point. Therefore, such derivatives can be used to answer the 'what-if' questions in a design environment. This important feature of the derivatives leads to the development of a discipline, called sensitivity analysis, which seeks efficient computational methods to compute the derivatives. Many works in reliability engineering, particularly in reliability-based design optimization, have given special attention to study the related derivatives [1-9]. However, some of these derivatives [2] are computed according to the functional relationship defined by the limit-state equation. Such derivatives are called behavior derivatives [10,11]. Since the MPP can be obtained as a result of an optimization process, the probabilistic derivatives can also be viewed as the derivatives of the optimal solution or optimum sensitivity derivatives [10,11]. In other words, the computation of such derivatives should involve not only the function of the limit-state equation but also the Kuhn-Tucker Necessary Conditions at the MPP.

The main goal of this paper is thus to present a procedure that computes the probabilistic derivatives as derivatives of an optimal solution. Furthermore, examples are used to demonstrate the applications of such optimum sensitivity derivatives to form better procedures for reliability analysis and reliability-based design optimization.

The paper is organized into three main parts. The first part, Sections II and III, reviews the procedures to derive optimum sensitivity derivatives and to perform the reliability analysis. The second part, Sections IV and V, derives the sensitivities of the results of reliability analysis with which a new MPP-based reliability analysis method is established. Numerical verifications are given in Section VI where simple academic examples are used to demonstrate the use of the newly devised method for reliability analysis and design optimization.

## II. DERIVATIVES OF CONSTRAINED OPTIMAL SOLUTIONS

A typical engineering optimization problem seeks the solution, which achieves the best performance among a given set of problem parameters within the limits of the available resources. The best performance is usually measured by a targeted objective, while the limited resources are usually prescribed by constraints. Mathematically, an optimization problem can be symbolically represented as [12]

$$\begin{array}{ll}
\min_{\mathbf{b} \in R^n} f(\mathbf{b}, p) & \\
\text{subject to} & \left. \begin{array}{l} h_i(\mathbf{b}, p) = 0, \quad i = 1, 2, \dots, m_h \\ g_j(\mathbf{b}, p) \leq 0, \quad j = 1, 2, \dots, m_g \end{array} \right\} \quad (\text{P.1})
\end{array}$$

where  $\mathbf{b} \in R^n$  is the design variable vector which is unknown,  $p$  is the given problem parameter,  $f(\mathbf{b}, p)$  is the objective and  $h_i(\mathbf{b}, p)$  and  $g_j(\mathbf{b}, p)$  are the equality and inequality constraints, respectively. The integers,  $m_h$  and  $m_g$ , represent the numbers of equality and inequality constraints, respectively. Note that the problem parameter can be a vector; however, for simplicity, it is taken as a scalar here.

Let the optimal solution of Problem (P.1) be set as  $\mathbf{b}^* \in R^n$ . It is then known [12] that,  $\mathbf{b}^*$ , satisfies the Kuhn-Tucker Necessary Conditions,

$$\frac{dL}{d\mathbf{b}}(\mathbf{b}^*, p) = 0 \quad (1)$$

$$h_i(\mathbf{b}^*, p) = 0, \quad i = 1, 2, \dots, m_h \quad (2)$$

$$g_j(\mathbf{b}^*, p) \leq 0, \quad j = 1, 2, \dots, m_g \quad (3)$$

$$\lambda_j g_j(\mathbf{b}^*, p) = 0, \quad j = 1, 2, \dots, m_g \quad (4)$$

$$\lambda_j \geq 0, \quad j = 1, 2, \dots, m_g \quad (5)$$

where  $L$  is the Lagrangian, defined as

$$L = f + \sum_i^{m_h} r_i h_i + \sum_j^{m_g} \lambda_j g_j \quad (6)$$

and the Lagrange multipliers,  $r_i$  and  $\lambda_j$  are uniquely defined if the gradients of the objective and the constraints are linearly independent at  $\mathbf{b}^*$ . According to Eqs. (3-5), the inequality constraints at the optimal solution,  $\mathbf{b}^*$ , can be classified into two possibilities; tight or non-tight. The tight constraints follow the relations

$$g_j(\mathbf{b}^*, p) = 0, \quad j \in \overline{m}_g \quad (7)$$

$$\lambda_j \geq 0, \quad j \in \overline{m}_g \quad (8)$$

while the non-tight ones follow a different set of relations

$$g_i(\mathbf{b}^*, p) < 0, \quad j \notin \overline{m}_g \quad (9)$$

$$\lambda_j = 0, \quad j \notin \overline{m}_g \quad (10)$$

where  $\overline{m}_g$  is a collection of tight constraints. Consequently, the Lagrangian of Eq. (6) can be simplified as

$$L = f + \sum_i^{m_h} r_i h_i + \sum_{j \in \overline{m}_g} \lambda_j g_j \quad (11)$$

The solution of an optimization problem should then include  $\mathbf{b}^*$  as well as the Lagrange multipliers as unknowns. Note that the Lagrange multipliers associated with tight inequality constraints are subjected to sign restriction, as indicated by Eq. (8), whereas those associated with equality constraints are not. The Kuhn-Tucker Necessary Conditions clearly show that the optimal solutions,  $\mathbf{b}^*$  and the Lagrange multipliers, will be changed if  $p$  is changed. Thus, the optimal solutions are functions of problem parameter,  $p$ , and their derivatives with respect to  $p$  are called optimum sensitivity derivatives.

### II.1 Optimum Sensitivity Derivative

An optimum sensitivity derivative is the limit of the difference between two neighboring optimal solutions that satisfy respective Kuhn-Tucker Necessary Conditions with different  $p$ 's. Therefore, the optimum sensitivity derivatives can be obtained by differentiating the Kuhn-Tucker Necessary Conditions, Eqs. (1-2) and Eqs. (7, 11) with respect to  $p$  as

$$\frac{\partial^2 L}{\partial \mathbf{b}^2} \frac{d\mathbf{b}^*}{dp} + \sum_i^{m_h} \frac{\partial h_i}{\partial \mathbf{b}} \frac{dr_i}{dp} + \sum_{j \in \overline{m}_g} \frac{\partial g_j}{\partial \mathbf{b}} \frac{d\lambda_j}{dp} + \frac{\partial^2 L}{\partial \mathbf{b} \partial p} = 0 \quad (12)$$

$$\left( \frac{\partial h_i}{\partial \mathbf{b}} \right)^T \frac{d\mathbf{b}^*}{dp} + \frac{\partial h_i}{\partial p} = 0, \quad i = 1, 2, \dots, m_h \quad (13)$$

$$\left( \frac{\partial g_j}{\partial \mathbf{b}} \right)^T \frac{d\mathbf{b}^*}{dp} + \frac{\partial g_j}{\partial p} = 0, \quad j \in \overline{m}_g \quad (14)$$

where the Hessian of  $L$  and  $\frac{\partial^2 L}{\partial \mathbf{b} \partial p}$  are given as

$$\frac{\partial^2 L}{\partial \mathbf{b}^2} = \frac{\partial^2 f}{\partial \mathbf{b}^2} + \sum_i^{m_h} r_i \frac{\partial^2 h_i}{\partial \mathbf{b}^2} + \sum_{j \in \overline{m}_g} \lambda_j \frac{\partial^2 g_j}{\partial \mathbf{b}^2} \quad (15)$$



$$\frac{\partial^2 L}{\partial \mathbf{b} \partial p} = \frac{\partial^2 f}{\partial \mathbf{b} \partial p} + \sum_i^{m_h} r_i \frac{\partial^2 h_i}{\partial \mathbf{b} \partial p} + \sum_{j \in \overline{m}_g} \lambda_j \frac{\partial^2 g_j}{\partial \mathbf{b} \partial p} \quad (16)$$

For clarification, Eqs. (12-14) can be put in a matrix form as

$$\begin{bmatrix} \frac{\partial^2 L}{\partial \mathbf{b}^2} & \frac{\partial h}{\partial \mathbf{b}} & \frac{\partial g}{\partial \mathbf{b}} \\ \left(\frac{\partial h}{\partial \mathbf{b}}\right)^T & 0 & 0 \\ \left(\frac{\partial g}{\partial \mathbf{b}}\right)^T & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{b}^*}{dp} \\ \frac{dr}{dp} \\ \frac{d\lambda}{dp} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 L}{\partial \mathbf{b} \partial p} \\ \frac{\partial h}{\partial p} \\ \frac{\partial g}{\partial p} \end{bmatrix} \quad (17)$$

where the summation notations in Eq. (17) are dropped for simplicity. Note that such differentiability is valid only when the tight constraint set remains the same in the neighborhood of  $p$ . In other words, Eq. (17) is valid only if the equations and the numbers of the tight constraints and their associated Lagrange multipliers as prescribed by Eqs. (7-8) remain unchanged around  $p$ . Otherwise, the optimal solutions  $\mathbf{b}^*$  and  $\lambda$  are not differentiable with respect to the parameter at the given value of  $p$ .

The dimension of the leading coefficient matrix in Eq. (17) is  $m = m_h + \overline{m}_g$  and the unknowns are  $d\mathbf{b}^*/dp$ ,  $dr/dp$  and  $d\lambda/dp$  for a single parameter,  $p$ . Though Eq. (17) has a unique solution as the leading coefficient matrix is non-singular, it is difficult to construct for general engineering applications. This is because its leading coefficient matrix involves second-order derivatives of  $L$ , such as Eqs. (15-16). Nevertheless, one particular type of the optimum sensitivity derivative, the derivatives of the objective, is much easier to obtain than the unknowns of Eq. (17). The derivation of such derivatives follows.

## II.2 Optimum Sensitivity Derivatives of the Objective

Differentiation of the objective at  $\mathbf{b}^*$  with respect to  $p$  yields

$$\frac{df}{dp} = \frac{\partial f}{\partial p} + \left(\frac{\partial f}{\partial \mathbf{b}}\right)^T \frac{d\mathbf{b}^*}{dp}$$

The derivative,  $\partial f / \partial \mathbf{b}$ , in the above equation can be replaced by the derivatives of the constraints, as stated by Eqs. (1, 11),

$$\frac{df}{dp} = \frac{\partial f}{\partial p} - \sum_i^{m_h} r_i \left( \frac{\partial h_i}{\partial \mathbf{b}} \right)^T \frac{d\mathbf{b}^*}{dp} - \sum_{j \in \overline{m}_g} \lambda_j \left( \frac{\partial g_j}{\partial \mathbf{b}} \right)^T \frac{d\mathbf{b}^*}{dp} \quad (18)$$

The terms associated with  $d\mathbf{b}^* / dp$  in the right-hand side of Eq. (18) can be replaced by the derivatives of the tight constraints. That is, according to Eqs. (13-14) and Eq. (11), the end result is given as

$$\frac{df}{dp} = \frac{\partial f}{\partial p} + \sum_i^{m_h} r_i \frac{\partial h_i}{\partial p} + \sum_{j \in \overline{m}_g} \lambda_j \frac{\partial g_j}{\partial p} = \frac{\partial L}{\partial p} \quad (19)$$

Equation (19) states that the derivative of the optimum objective with respect to a parameter is equal to the derivative of the Lagrangian with respect to the same parameter at the optimal solution. The equation gives a first order approximation of the change in the optimum objective due to the change in the problem parameter.

### III. RELIABILITY ANALYSIS

Given a response condition of random variables,  $\mathbf{X}$ , reliability analysis is interested in finding the probability of failure of such a condition. For example, a constraint is defined by the response condition

$$G(\mathbf{X}) \leq 0 \quad (20)$$

which gives the probability of failure as

$$P_f = P(G(\mathbf{X}) > 0) \quad (21)$$

The reliability is then given by

$$R = 1 - P_f \quad (22)$$

The limit-state equation is defined by the failure surface, which is given by

$$G(\mathbf{X}) = 0. \quad (23)$$

Note that in this study, the random variables,  $\mathbf{X}$ , are assumed to be statistically independent and normally distributed. If they are not, one should take proper procedures to find their suitable equivalences [2] before reliability analysis.

One approach to solve Eq. (21) is the approximate probability integration method whose goal is to find a rotationally invariant reliability measurement, with which First-Order Reliability Method (FORM) or Second-Order Reliability Method (SORM) is developed. Most of these methods may be classified into two groups; the Reliability Index Approach (RIA) and the

Performance Measure Approach (PMA) [6, 7]. As an example, the HL-RF algorithm [13-16] is a method of RIA, whereas the AMV+ [1, 17] is one of PMA.

In RIA, the objective is to find the first-order safety reliability index,  $\beta$ , which is equal to the shortest distance between the origin of the  $\mathbf{U}$ -space and the failure surface. The contact point on the failure surface is called the Most Probable Point (MPP). The  $\mathbf{u}$  is the reduced variable vector, whose component is defined by

$$u_i = \frac{X_i - \mu_i}{\sigma_i}$$

where  $\mu_i$  and  $\sigma_i$  are the mean and the standard deviation of the corresponding  $X_i$ . The limit-state equation is thus rewritten as  $G(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\mu}) = 0$ . Mathematically, the reliability index can be viewed as the objective of an optimization problem with the limit-state equation as the equality constraint,

$$\text{subject to } \left. \begin{array}{l} \min_{\mathbf{u} \in \mathbb{R}^n} \|\mathbf{u}\| \\ G(\mathbf{u}) = 0 \end{array} \right\} \quad (\text{P.2})$$

where the reduced variable vector,  $\mathbf{u}$ , is considered as the design variable vector. At its optimal solution,  $\mathbf{u}^*$ , one finds the reliability index,  $\beta = \|\mathbf{u}^*\|$ , which yields the first-order probability of failure as  $P_f = \Phi(-\beta)$ .

In PMA, the objective is to compute the first-order probabilistic performance measure,  $G_p^*$ . It is defined as the offset of the performance,  $G(\mathbf{X})$ , so that the shortest distance between the limit-state equation,  $G(\mathbf{X}) - G_p^* = 0$  and the origin of the  $\mathbf{U}$ -space is equal to a given target reliability service,  $\Phi(-\beta_o)$ .  $G_p^*$  can also be found as the smallest value of  $G$  that is tangent to the targeted reliability surface, represented by a sphere constraint,  $\|\mathbf{u}\| = \beta_o$ . Mathematically, the first-order probabilistic performance measure is obtained as the objective of an optimization problem as

$$\text{subject to } \left. \begin{array}{l} \min_{\mathbf{u} \in \mathbb{R}^n} G(\mathbf{u}) \\ \|\mathbf{u}\| = \beta_o \end{array} \right\} \quad (\text{P.3})$$

where the constraint requires the solution to achieve the targeted reliability index.

The optimization formulation of the RIA is very similar to that of the PMA. In fact, at their optimal solutions, their Kuhn-Tucker Necessary Conditions yield similar relations;

for the RIA,

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} + \lambda_r \frac{\partial G}{\partial \mathbf{u}} = 0 \quad (24)$$

and, for the PMA,

$$\lambda_p \frac{\mathbf{u}}{\|\mathbf{u}\|} + \frac{\partial G}{\partial \mathbf{u}} = 0 \quad (25)$$

It is noted that their Lagrange multipliers can be either positive or negative, since they correspond to equality constraints. Furthermore, their Lagrange multipliers exhibit a reciprocal relationship, if  $\beta^*$  is the same for both problems. However, their optimal solutions,  $\mathbf{u}^*$ , are usually not the same, unless the targeted reliability index in PMA is exactly the same as the reliability index obtained in RIA. Since  $\|\mathbf{u}^*\| = \beta^*$  in RIA and  $\|\mathbf{u}^*\| = \beta_0$  in PMA, Eqs. (24-25) give convenient means to compute the respective Lagrange multipliers as

$$\lambda_r = \frac{-\beta^*}{\left(\frac{\partial G}{\partial \mathbf{u}}\right)^T \mathbf{u}^*} \quad (26)$$

and

$$\lambda_p = \frac{-\left(\frac{\partial G}{\partial \mathbf{u}}\right)^T \mathbf{u}^*}{\beta_0} \quad (27)$$

#### IV. SENSITIVITIES OF RELIABILITY ANALYSIS RESULTS

Since the standard reliability methods, RIA or PMA, are all formulated as constrained optimization problems, Eqs. (17) and (19) can be directly applied here to calculate their optimum sensitivity derivatives. Note that in Problems (P.2) and (P.3), the reduced variables,  $\mathbf{u}$ , are treated as the design variables, whereas the standard deviations  $\sigma$ , the mean values  $\mu$  of the random variables or the targeted  $\beta_0$  in PMA are treated as the problem parameters. Furthermore, the optimum sensitivity derivatives of Problems (P.2) and (P.3) are readily available, as they involve only equality constraints.

##### IV.1 Problem (P.2) for RIA

The sensitivity of the reliability index,  $\beta^*$ , with respect to a problem parameter can be obtained from Eq. (19) as

$$\frac{d\beta^*}{dp} = \frac{\partial \beta^*}{\partial p} + \lambda_r \frac{\partial G}{\partial p} = \lambda_r \frac{\partial G}{\partial p} \quad (28)$$

where  $\partial\beta^*/\partial p = 0$  since  $\beta^*$  is a function of  $\mathbf{u}^*$  only. To find the sensitivity derivatives of the optimum reduced variables,  $\mathbf{u}^*$ , an equation of the second order derivatives of  $\beta^*$  and the constraint,  $G(\mathbf{u}) = 0$ , is then required, as indicated by Eq. (17),

$$\begin{bmatrix} \left( \frac{I}{\beta^*} - \frac{\mathbf{u}^* \mathbf{u}^{*T}}{\beta^{*3}} \right) + \lambda_r \frac{\partial^2 G}{\partial \mathbf{u}^2} & \frac{\partial G}{\partial \mathbf{u}} \\ \left( \frac{\partial G}{\partial \mathbf{u}} \right)^T & 0 \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{u}^*}{dp} \\ \frac{d\lambda_r}{dp} \end{bmatrix} = - \begin{bmatrix} \lambda_r \left( \frac{\partial^2 G}{\partial \mathbf{u} \partial p} \right) \\ \frac{\partial G}{\partial p} \end{bmatrix} \quad (29)$$

The random variable,  $X_i$ , is related to its reduced form,  $u_i$ , as  $X_i = u_i \sigma_i + \mu_i$ . Thus, the components of the gradient and the Hessian of  $G(\mathbf{u})$  or  $G(\mathbf{X}) = 0$  in Eq. (29) are given as

$$\frac{\partial G}{\partial \mathbf{u}} = \left\{ \frac{\partial G}{\partial u_i} \right\} = \left\{ \frac{\partial G}{\partial X_i} \sigma_i \right\}$$

and

$$\frac{\partial^2 G}{\partial \mathbf{u}^2} = \left[ \frac{\partial^2 G}{\partial u_i \partial u_j} \right] = \left[ \frac{\partial^2 G}{\partial X_i \partial X_j} \sigma_i \sigma_j \right]$$

#### IV.2 Problem (P.3) of PMA

The objective function  $f$  and the equality constraint in this case are  $G(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\mu})$  and  $h(\mathbf{u}, \beta_0) = (\mathbf{u}^T \mathbf{u})^{1/2} - \beta_0 = 0$ , respectively. The derivatives of the first-order probabilistic performance measure,  $G_p^*$ , with respect to the problem parameters are given by Eq. (19) as

$$\frac{dG_p^*}{dp} = \frac{\partial G_p^*}{\partial p} + \lambda_p \frac{\partial h}{\partial p} \quad (30)$$

In particular, if the standard deviations and the mean values of random variables are considered as the problem parameters, then Eq. (30) is simplified as

$$\frac{dG_p^*}{dp} = \frac{\partial G_p^*}{\partial p} \quad (31)$$

as the constraint,  $h$ , is a function of  $\mathbf{u}$  and  $\beta_0$  only. On the other hand, if the target  $\beta_0$  is selected as the problem parameter, the first term,  $\partial G_p^* / \partial p$ , becomes zero and the second term,  $\partial h / \partial p$ , equals -1. As a result, Eq. (30) is reduced to

$$\frac{dG_p^*}{d\beta_0} = -\lambda_p \quad (32)$$

The sensitivity derivatives of the optimum solution,  $\mathbf{u}^*$ , of the PMA with respect to the problem parameters are given by Eq. (17) as

$$\begin{bmatrix} \frac{\partial^2 G}{\partial \mathbf{u}^2} + \frac{\lambda_p}{\beta_0} \mathbf{I} & \frac{\mathbf{u}^*}{\beta_0} \\ \left( \frac{\mathbf{u}^*}{\beta_0} \right)^T & 0 \end{bmatrix} \begin{Bmatrix} \frac{d\mathbf{u}^*}{dp} \\ \frac{d\lambda_p}{dp} \end{Bmatrix} = - \begin{Bmatrix} \frac{\partial^2 G}{\partial \mathbf{u} \partial p} + \lambda_p \frac{\partial^2 h}{\partial \mathbf{u} \partial p} \\ \frac{\partial h}{\partial p} \end{Bmatrix} \quad (33)$$

where the relations,  $\partial h / \partial \mathbf{u} = \mathbf{u}^* / \beta_0$  and  $\partial^2 h / \partial \mathbf{u}^2 = \mathbf{I} / \beta_0$ , have been used in derivation.

## V. APPLICATIONS

The optimum sensitivity derivatives derived above provide means to support the development of a better algorithm for reliability analysis as well as reliability-based design optimization (RBDO).

### V.1 Gradients for RBDO

A reliability-based design optimization usually involves random variables as design variables and reliability index or performance measure, depending upon its formulation, as its objective or constraints. The randomness of a design variable in a reliability-based design optimization is usually represented by its mean and standard deviation. Therefore, the mean and the standard deviation of the random variables can be directly modeled as the design variables. In this case, Eq. (28) or Eq. (31) provides necessary derivatives of the reliability index or the performance measure with respect to the mean or the standard deviation of a random variable to support any reliability-based design optimization algorithm. More specifically, in terms of the random variable,  $X_i$ , the derivatives of the reliability index of Eq. (28) with respect to the mean and the standard deviation are rewritten, respectively, as

$$\frac{d\beta^*}{d\mu_i} = \lambda_r \frac{\partial G}{\partial X_i} \quad (34)$$

and

$$\frac{d\beta^*}{d\sigma_i} = \lambda_r \frac{\partial G}{\partial X_i} u_i^* \quad (35)$$

On the other hand, the derivatives of performance measure of Eq. (31) with respect to the mean and the standard deviation are given as

$$\frac{dG_p^*}{d\mu_i} = \frac{\partial G}{\partial X_i} \quad (36)$$

and

$$\frac{dG_p^*}{d\sigma_i} = \frac{\partial G}{\partial X_i} u_i^* \quad (37)$$

## V.2 Search Algorithms for Reliability Analysis

Equations (24-25) provide an important rule to guide the search of the reliability index or performance measurement that  $\mathbf{u}^*$  is proportional to  $\partial G / \partial \mathbf{u}$ . Thus, the unit direction of  $\mathbf{u}^*$  is given as

$$\mathbf{n} = \frac{\partial G / \partial \mathbf{u}}{\|\partial G / \partial \mathbf{u}\|} \quad (38)$$

In PMA, since the length of vector  $\mathbf{u}^*$  is limited to be the given  $\beta_0$ , as prescribed in the constraint, a very simple recursive formula can then be devised to find the  $\mathbf{u}^*$  as

$$\mathbf{u}^{i+1} = \beta_0 \mathbf{n}^i \quad (39)$$

This algorithm is quite efficient. However, Choi and Youn experienced convergence difficulty of Eq. (39) when the limit-state equation exhibits concavity [7]. They then replaced the current unit direction,  $\mathbf{n}^i$ , in Eq. (39) by the average of the last three consecutive unit directions. Their investigation can be found in Reference 7.

The search algorithm in RIA is more complicated than that in PMA, because  $\beta^*$  is not known in advance. Equation (39) is still valid; however, a correction step is needed to improve the estimate of  $\beta$ . A recursive formula on  $\mathbf{u}$  as well as  $\beta$  can then be set up as

$$\mathbf{u}^{i+1} = \beta^{i+1} \mathbf{n}^i \quad (40)$$

where  $\beta^{i+1}$  can be obtained through the Newton-Raphson's iteration on the constraint,  $G(\mathbf{u}^{i+1}) = G(\beta^{i+1} \mathbf{n}^i) = 0$ , as

$$\left( \left( \frac{\partial G}{\partial \mathbf{u}} \right)^T \mathbf{n}^i \right) \Delta \beta^i = -G(\beta^i \mathbf{n}^i) \quad (41)$$

Once  $\beta^{i+1}$  is converged,  $\mathbf{u}^{i+1}$  is then found by Eq. (40). One can then move to next iteration. This constitutes the basic algorithm of Hasofer-Lind method [13-14]. An improved version [2,15,16] of the Hasofer-Lind method is to reduce computational burden of the Newton-Raphson's iteration by updating  $\mathbf{u}$  and  $\beta$  in the same iteration as

$$\mathbf{u}^{i+1} = \beta^i \mathbf{n}^i + \Delta\beta^i \mathbf{n}^i \quad (42)$$

where  $\beta^i$  is obtained by

$$\beta^i = \mathbf{u}^{iT} \mathbf{n}^i$$

and  $\Delta\beta^i$  is obtained by solving Eq. (41) as

$$\Delta\beta^i = \frac{-G(\beta^i \mathbf{n}^i)}{\left(\frac{\partial G}{\partial \mathbf{u}}\right)_i^T \mathbf{n}^i}$$

These RIA methods, however, fail to converge, if the limit-state function is concave near the search point [7]. A more robust method for RIA is proposed hereafter, which is derived based on the PMA algorithm and its optimum sensitivity equation, Eq. (32).

### V.3 An Alternative Algorithm for RIA

The key motivation of this new algorithm is the observation that the target  $\beta_0$  of PMA is identical to the reliability index,  $\beta^*$ , of RIA if the performance measure,  $G_p^*$ , in PMA reaches zero value. To achieve a zero  $G_p^*$ , the new algorithm uses Eq. (32) to estimate the amount of the change in  $\beta_0$  needed to reduce the non-zero value of  $G^*$ . Eq. (32) gives an estimate,

$$\Delta G^* = -\lambda_p \Delta\beta \quad (43)$$

Let  $\Delta G^* = 0 - G_p^*$  be the gap between zero and the current value of  $G_p^*$ . The change in target  $\beta_0$  that is required to achieve a zero  $G_p^*$  is then estimated by Eq. (43) as

$$\Delta\beta = G_p^* / \lambda_p \quad (44)$$

The updated reliability index,  $\beta + \Delta\beta$ , will yield a new  $G_p^*$  that is closer to zero, at least in the first order sense. Repeated use of Eq. (44) can lead the PMA search to  $G_p^* = 0$ . In short, the new algorithm can be summarized as follows:

Step 1: Start with an initial target  $\beta^i = \beta_0$  and initial values of  $\mathbf{u}^i = \mathbf{u}_0$ .

Step 2: Follow the PMA procedure [7] to obtain the converged performance measure,  $G_p^{*i}$ .

Step 3: Compute  $\lambda_p$  by Eq. (27), which is evaluated at the converged  $\mathbf{u}^i$  and  $\beta^i$  in Step 2.



Step 4: Update  $\beta^{i+1} = \beta^i + \Delta\beta^i$ , where  $\Delta\beta^i = G_p^{*i} / \lambda_p$ .

Step 5: Return to Step 2, until  $\Delta\beta$  and  $G_p^*$  achieve the required tolerances for convergence.

The above algorithm is similar to the one presented by Du and Chen [18]. Both trace the MPP locus to locate the  $\beta^*$ . Nevertheless, the former follows the tangential direction of the MPP locus, whereas the latter follows an extrapolated MPP locus. A MPP locus, represented by the curve of  $G_p^*(\beta)$ , is shown in Fig. 1. It is also important to note that Eq. (44) is in the same form as Eq. (41). Nevertheless, Eq. (44) is only valid at the optimum solution of the PMA. Therefore, it represents a slope on the MPP locus, whereas Eq. (41) does not.

Since  $\beta$  is restricted to be positive, the  $G_p^*(\beta)$  curve does exhibit a discontinuity at  $\beta = 0$ , as shown in Fig. 1. Fortunately, the probabilities of failure for most of applications are pertaining to the tip of either the upper or the lower branch of the curve, so as to avoid the discontinuity. The detailed discussion of this matter is given in the next section.

#### V.4 Infeasible Region: $G(X) \leq 0$ or $G(X) \geq 0$

In a classical design optimization problem, a feasible design is usually defined by  $G(X) \leq 0$ , which is associated with a failure probability of  $P_f(G(X) > 0)$ . The PDF plot and the MPP locus of this case are shown in Fig. 2. However, in a reliability-based design optimization, a reliable constraint is usually defined by  $G(X) \geq 0$  [2], which is associated with a failure probability of  $P_f(G(X) < 0)$ . This case is shown in Fig. 3. Note that  $P_f(G(X) < 0)$  occurs on the far left tail of the probability distribution of  $G(u)$  along the direction of  $u$  in the standard normal space, whereas  $P_f(G(X) > 0)$  on the far right tail. This inconsistency, also noted by Padmanabhan and Batill [19], can cause confusion in selecting a proper solution procedure for reliability analysis. For example, the reliability analysis formulation and procedure derived in Sections V.2 and V.3 are based upon the definition of failure probability,  $P_f(G(X) > 0)$ . Modifications of these formulation and procedure for the case of  $P_f(G(X) < 0)$  are in order.

Note that both cases,  $P_f(G(X) > 0)$  and  $P_f(G(X) < 0)$ , enjoy the same limit-state equation,  $G(X) = 0$ . Therefore, both failure probabilities can be calculated from the same reliability index. Since the reliability index is always positive as defined by  $\beta^* = \|u\|$ , the recursive equation of Eq. (39) or (40) are valid only when  $u$  and  $n$  are in the same direction, where  $n$  is defined by Eq. (38) as the steepest ascending direction at the given point on the  $G(u)$  curve. In the case of  $P_f(G(X) > 0)$ ,  $u$  is indeed in the same direction of  $n$ , as shown in Fig. 4, whereas in the case of  $P_f(G(X) < 0)$ ,  $u$  is not, as shown in Fig. 5. Thus, the basic equation, Eq. (39) or (40), is valid only for the case of  $P_f(G(X) > 0)$ . Otherwise, a negative sign should be added to these equations. For example, Eq. (40) should be modified as

$$u^{i+1} = -\beta^{i+1} n^i \quad (45)$$

It then follows that  $\lambda_r$  and  $\lambda_p$  are positive for the case of  $P_f(G(X) < 0)$  and negative for the case of  $P_f(G(X) > 0)$ . Consequently, the slope of the MPP locus,  $G_p^*(\beta)$ , is negative for the former

case and positive for the latter, as indicated by Eq. (32). Figures 2.b and 3.b demonstrate the slopes of the MPP locus for both cases.

The difference in the slopes of  $G_p^*(\beta)$  has an important implication in the formulation of reliability constraints. For example, if a reliability constraint requires the failure probability of constraint,  $G(\mathbf{X}) \geq 0$ , to be less than a targeted value,  $P_0$ , the reliability constraint can be stated as

$$P_f(G(\mathbf{X}) < 0) \leq P_0 \quad (46)$$

The above constraint is equivalent to

$$\beta_0 \leq \beta^* \quad (47)$$

where the reliability index,  $\beta^*$ , is calculated by  $\Phi(-\beta^*) = P_f(G(\mathbf{X}) < 0)$  and the targeted  $\beta_0$  is calculated by  $\Phi(-\beta_0) = P_0$ . Eq. (47) implies, according to Fig. 3.b,

$$0 \leq G_p^*(\beta_0). \quad (48)$$

On the other hand, if the reliability constraint requires

$$P_f(G(\mathbf{X}) > 0) \leq P_0. \quad (49)$$

The constraint is equivalent to, in terms of reliability indices,

$$\beta_0 \leq \beta^* \quad (50)$$

which is the same as Eq. (47), or, in terms of performance measurement, according to Fig. 2.b,

$$G_p^*(\beta_0) \leq 0. \quad (51)$$

In the RIA-based RBDO, reliability constraints are expressed in the form of either Eq. (47) or (50), whereas in the PMA-based RBDO, reliability constraints are expressed in the form of either Eq. (48) or (51).

## VI. NUMERICAL EXAMPLES

Numerical studies are conducted in this section to verify the equations derived in Sections IV and V and to demonstrate their applications to reliability analysis and reliability-based design optimization. The example limit-state equations used in this study are the ones studied in Ref. 7. One of them is a convex function, and the other is a concave one. Reference 7 reported that the HL-RF method failed to converge in both examples and the AMV+ method performed very well in the convex example, but failed to converge in the concave one. A hybrid method was then proposed in Ref. 7 that is workable for both cases. If the limit-state equation is

convex, the hybrid method will be the same as the AMV+ method, which will take the gradient of  $G(X)$  at the current design point as the search direction. If the limit-state equation is concave, the hybrid method will take an average of the gradients evaluated at the last three consecutive design points as the new search direction.

The performance of the new RIA developed in Section V.3 for reliability analysis is first examined in Section VI.1. The new RIA first uses the hybrid method [7] to find the performance measure,  $G_p^*$ , for the targeted  $\beta_0$ . If the value of  $G_p^*$  does not converge to zero, the hybrid method is then restarted to find the  $G_p^*$  of the new  $\beta_0$ , updated by Eq. (44). The process is continued until  $G_p^*$  reaches zero within a tolerance. Once convergence, the derivatives of the results of reliability analysis are readily available, according to Eqs. (28) and (31), to support RBDO.

In Section VI.2, two approaches are first considered for RBDO. One is a RIA-based RBDO which models its reliability constraints in the form of Eq. (47) or (50), the other is a PMA-based RBDO which models its reliability constraints in the form of Eq. (48) or (51). The new RIA is incorporated with a gradient-based optimization algorithm, called Linearization Method [20, 21] to form the RIA-based RBDO solution procedure. On the other hand, the hybrid method is incorporated with the Linearization Method to form the PMA-based RBDO. The performance of the RIA-based RBDO will then be compared with that of a PMA-based RBDO. It is found that the RIA-based RBDO takes more function and gradient evaluation for reliability analysis than the PMA-based RBDO does. However, the former is much more robust than the latter in the line search part of a design optimization procedure. This observation leads to the development of a new RBDO procedure that incorporates the concept of active reliability constraint to achieve better computational efficiency in RBDO.

The suggested method will start with the hybrid method to calculate  $G_p^*$  for the targeted  $\beta_0$  for each of the reliability constraints. If  $G_p^*$  is less than a given margin, the constraint is judged as active-in-reliability. That means, the current design point is near the boundary of the reliability constraint. The new RIA will only be called upon to calculate the reliability indices of those active-in-reliability constraints. Once the values of such constraints are computed, the active constraint set can then be determined to guide the search direction in an optimization algorithm. Note that two different active constraint sets are mentioned here. One is active in reliability analysis, and the other is active in design optimization. The former set is bigger than the latter.

### VI.1 The New RIA and Optimum Sensitivities

The new RIA is first applied to find the reliability index of the limit-state equation,

$$G_1(X_1, X_2) = -e^{(X_1-1)} - X_2 + 10 \quad (52)$$

which is a convex function. The random variables,  $X_1$  and  $X_2$  are both normal and given as  $X_1 \sim N(6.0, 0.8)$  and  $X_2 \sim N(6.0, 0.8)$ . The targeted  $\beta_0$  is set to be 3.0 and the initial values of  $X_1$  and  $X_2$  are assigned to be 6 to start reliability analysis. The new RIA is called converged, if the

changes of the performance measurement,  $G_p^*$  and the reliability index,  $\beta^*$ , are, respectively, less than 0.0005 and 0.001 between consecutive iterations. The results are summarized in Table 1. It is noted that the new RIA takes 7 iterations to find the performance measure,  $G_p^* = -0.357$ , for a  $\beta_0$  of 3 and it takes additional 5 iterations to reach the converged reliability index,  $\beta^* = 2.878$ . The probability of failure,  $P_f (G_I < 0)$ , is then calculated as 0.201%, which is in a good agreement with 0.216%, the probability of failure calculated by Monte Carlo simulation with 200,000 samples. Note that each of iterations reported in the table involves a function and a gradient evaluation.

Table 1: Convergence History of the New RIA – Example 1

Iteration No.	$X_1$	$X_2$	$G_I$	$\beta$	$\lambda_p$
1	6.828	8.252	0.905	3.0	0
2	7.546	7.835	0.437	3.0	0.9835
3	8.076	7.202	-0.138	3.0	1.5019
4	8.271	6.773	-0.341	3.0	2.4332
5	8.310	6.647	-0.357	3.0	2.9593
6	8.317	6.624	-0.357	3.0	3.0734
7	8.318	6.620	-0.357	3.0	3.0918
-----					
1	8.228	6.596	-1.413-2	2.884	3.2189
2	8.214	6.647	-1.669-2	2.884	2.8480
3	8.211	6.656	-1.677-2	2.884	2.8110
-----					
1	8.206	6.565	-8.358-5	2.878	2.8104
2	8.206	6.659	-9.260-5	2.878	2.7917

The second example limit-state equation studied here is a concave function,

$$G_2(X_1, X_2) = [e^{(0.8X_1 - 1.2)} + e^{(0.7X_2 - 0.6)} - 5]/10 \quad (53)$$

where  $X_1$  and  $X_2$  are normally distributed random variables and  $X_1 \sim N(4.0, 0.8)$  and  $X_2 \sim N(5.0, 0.8)$ . The targeted  $\beta_0$  is set to be 3.0. Again, the convergence tolerances are taken as 0.0005 and 0.001, respectively, for  $G_p^*$  and  $\Delta\beta$ . The results are summarized in Table 2. The new RIA takes 9 iterations to reach the converged  $G_p^*$  of 0.204 for a targeted  $\beta_0$  value of 3. Additional 7 iterations are needed for the method to find the converged reliability index,  $\beta^* = 3.803$ , which gives a probability of failure,  $P_f (G_2 < 0)$ , a value of 7.16E-3%. The probability of failure calculated by Monte Carlo simulation is 5.5 E-3% with 2 million samples.

Table 2: Convergence History of the New RIA – Example 2

Iteration No.	$X_1$	$X_2$	$G_2$	$\beta$	$\lambda_p$
1	2.989	2.823	0.225	3.0	0.000
2	2.348	3.259	0.234	3.0	0.290
3	3.073	2.786	0.238	3.0	0.305
4	2.539	3.096	0.209	3.0	0.286
5	2.710	2.976	0.204	3.0	0.302
6	2.561	3.068	0.207	3.0	0.299
7	2.765	2.942	0.205	3.0	0.302
8	2.677	2.998	0.204	3.0	0.297
9	2.690	2.989	0.204	3.0	0.300
-----					
1	2.368	2.549	2.714E-2	3.681	0.244
2	2.310	2.588	2.725E-2	3.681	0.244
-----					
1	2.341	2.450	9.676E-4	3.803	0.217
2	2.199	2.548	1.628E-3	3.803	0.212
3	2.412	2.405	3.003E-3	3.803	0.214
4	2.233	2.523	8.631E-4	3.803	0.210
5	2.287	2.486	4.513E-4	3.803	0.213

To verify the optimum sensitivities of the performance measurement and the reliability index, given by Eqs. (31) and (28), respectively, central differencing is conducted for the above two examples. In both cases, the mean value of  $X_1$ ,  $\mu_1$ , and the standard deviation of  $X_2$ ,  $\sigma_2$ , are considered as the design parameters. The perturbations of them are taken as 0.01 and 0.008, respectively, in finite differencing studies. The results summarized in Table 3 show that the analytical derivatives calculated by Eqs. (31) and (28) are in good agreement with those calculated by central finite differencing (C.D.).

## VI.2 Reliability-Based Design Optimization

A general reliability-based design optimization problem can be symbolically written as

$$\left. \begin{array}{l} \min_{\mu \in R^n} f(\mu) \\ \text{subject to} \\ P_f \leq P_o \end{array} \right\} \quad (\text{P.4})$$

Table 3a: Optimum Sensitivities of  $G_1$  and  $G_2$  with respect to  $\mu_l$ 

Case	$+\Delta\mu_l$	$-\Delta\mu_l$	C. D.	Analytical
$G_{1p}$	-0.3955	-0.3208	-3.7371	-3.7369
$G_{2p}$	0.2059	0.2018	0.2058	0.2072
$\beta_1^*$	2.8663	2.8902	-1.1967	-1.1964
$\beta_2^*$	3.8095	3.7956	0.6928	0.7042

Table 3b: Optimum Sensitivities of  $G_1$  and  $G_2$  with respect to  $\sigma_2$ 

Case	$+\Delta\sigma_2$	$-\Delta\sigma_2$	C. D.	Analytical
$G_{1p}$	-0.3642	-0.3518	-0.7785	-0.7762
$G_{2p}$	0.1976	0.2101	0.7834	0.7825
$\beta_1^*$	2.8759	2.8806	-0.2954	-0.2954
$\beta_2^*$	3.7771	3.8284	-3.2037	-3.2270

where the probability of failure,  $P_f$ , can be calculated by either  $P_f(G(\mathbf{X}) < 0)$  or  $P_f(G(\mathbf{X}) > 0)$ , depending upon the definition of failure.

The specific test problem studied here is also taken from Reference 7. The design variables are the means of two statistically independent and normally distributed variables;  $X_1 \sim N(5.0, 0.8)$  and  $X_2 \sim N(6.0, 0.8)$ , where the initial guesses of the design variables are set to be 5.0 and 6.0, respectively. The objective and the constraints of the problem are respectively given as

$$f(\boldsymbol{\mu}) = 3\mu_1^2 - 2\mu_1\mu_2 + 3\mu_2^2 \quad (54)$$

and

$$P_f(G_i(\mathbf{X}) < 0) \leq P_{i0}; \quad i = 1, 2, \quad (55)$$

where the limit-state equations,  $G_1$  and  $G_2$ , are defined by Eqs. (52) and (53), and  $P_{10}$  and  $P_{20}$  are the required probabilities.

In an RIA-based RBDO, the statements of failure probability, Eq. (55), are rewritten in the form of reliability index as indicated by Eq. (47),

$$\beta_{i0} - \beta_i^* \leq 0; \quad i = 1, 2 \quad (56)$$

whereas, in a PMA-based RBDO, they are rewritten in the form of performance measurements as indicated by Eq. (48),

$$-G_p^*(\beta_{i0}) \leq 0; \quad i = 1, 2 \quad (57)$$

The Linearization Method [20, 21] is used here as the optimization algorithm to search for the best design. The Linearization Method is a special method of Sequential Quadratic Programming Technique in which an approximate sub-problem is formed in each optimization iteration as

$$\left. \begin{array}{l} \min_{\delta \mathbf{b} \in R^n} \frac{1}{2} \delta \mathbf{b}^T \delta \mathbf{b} + \mathbf{l}_o^T \delta \mathbf{b} \\ \text{subject to} \quad h_i + \mathbf{l}_{h_i}^T \delta \mathbf{b} = 0; \quad i = 1 \text{ to } m_h \\ \quad \quad \quad g_j + \mathbf{l}_{g_j}^T \delta \mathbf{b} \leq 0, \quad j \in A \end{array} \right\} \quad (P.5) \quad (58)$$

where  $\delta \mathbf{b}$  is the change in design and  $\mathbf{l}_o$ ,  $\mathbf{l}_{h_i}$ , and  $\mathbf{l}_{g_j}$  are the gradients of the objective function, equality and inequality constraints, respectively. The constraint set,  $A$ , includes only those constraints that are  $\epsilon$ -active. That is, the constraint,  $g_j$ , that satisfies the condition,

$$g_j + \epsilon \leq 0 \quad (58)$$

The sub-problem, (P.5), is a convex problem that can be solved by a dual method. As a result, the computational effort of the method depends upon the number of the active constraints rather than the number of design variables. The solution of the sub-problem (P.5),  $\delta \mathbf{b}$ , gives a favored direction to reduce the cost as well the constraint violations. The new design point is then sought to be

$$\mathbf{b}^{i+1} = \mathbf{b}^i + \alpha \delta \mathbf{b}$$

where  $\alpha$  is the step size. The value of  $\alpha$  is selected to guarantee that a reduction in the merit function,  $\psi$ , is achieved by the new design. That is,

$$\psi(\mathbf{b}^{i+1}) = \psi(\mathbf{b}^i + \alpha \delta \mathbf{b}) < \psi(\mathbf{b}^i) \quad (59)$$

The merit function,  $\psi$ , is defined as

$$\psi = f + NF$$

where  $F$  is the maximal value of the violations and  $N$  is a positive value, selected to be greater than the sum of all the Lagrange multipliers,

$$N \geq \sum_{i=1}^{m_h} |r_i| + \sum_{j \in A} \lambda_j$$

where  $r_i$  and  $\lambda_j$  are the Lagrange multipliers corresponding to the equality and inequality constraints, respectively. Thus, Eq. (59) ensures that the new design will achieve a reduction in the cost as well as the constraint violations. It has been proven [21] that the Linearization Method reaches a local minimum if the  $L_2$ -norm of the search direction approaches to zero, i.e.,

$$\|\delta \mathbf{b}\| \leq \delta_o \quad (60)$$

The above equation serves conveniently as a convergence criterion.

The convergence histories of the PMA-based and the RIA-based RBDO are tabulated in Tables 4 and 5. Both methods drove the  $L_2$ -norm of the search direction to an acceptable level at the end of the optimization and they arrived at similar local minima. As expected, the PMA method is more efficient than the new RIA method for reliability analysis. The PMA-based method takes 131 and 39 function and gradient evaluations in total and the RIA-based method, 155 and 111, respectively, for reliability analysis of limit-state equations,  $G_1$  and  $G_2$ . However, the PMA-based method takes 336 and 96 function and gradient evaluations for the purpose of line search, whereas the RIA-based method takes only 6 and 8. The line search factor makes the RIA-based method a better one for overall performance for this particular RBDO example.

Note that each line search step in an optimization iteration requires a new reliability analysis. In the current study, the converged random variables,  $\mathbf{u}^*$ , of the current line search step are kept as the initial values to start the next line search step. Furthermore, the final  $\beta^*$  and  $\mathbf{u}^*$  at the end of the current optimization iteration are also kept as the initial values to start the next optimization iteration.

The traditional design optimization procedure does not work well for RBDO. This is because the traditional procedure requires completed reliability analyses before judgment can be made regarding the activeness of the constraints. For example, the second constraint in the last example is never involved in the design optimization process. However, the RIA-based RBDO still performs 119 function and gradient evaluations to eventually determine the second reliability constraint is indeed not active. To reduce such unnecessary computational burden, the



Table 4. Results of PMA-based RBDO

Iteration No.	Cost	$G_{p1}$	$G_{p2}$	$\mu_1$	$\mu_2$	No. of Analyses ( $G_{p1}/G_{p2}$ )	$\ \delta\ $	Step-Size $\alpha$	No. of Analyses for Line Search ( $G_{p1}/G_{p2}$ )
1	123.	0.971	1.429	5.000	6.000	9/4	0.2634	1.0	
2	114.81	0.779	1.676	4.850	5.783	9/3	0.2545	1.0	
3	107.17	0.618	1.908	4.704	5.575	9/3	0.2457	1.0	
4	100.00	0.481	2.126	4.562	5.375	9/3	0.2373	1.0	
5	93.40	0.366	2.333	4.423	5.182	9/2	0.2292	1.0	
6	87.21	0.267	2.530	4.288	4.997	9/2	0.2213	1.0	
7	81.43	0.182	2.178	4.157	4.828	7/2	0.2133	1.0	
8	76.04	0.110	2.899	4.0303	4.647	7/2	0.2066	1.0	
9	71.00	0.0469	3.071	3.906	4.481	7/2	0.1995	0.5	7/2
10	68.63	0.0188	3.153	3.845	4.402	7/2	0.1962	0.25	7/2+7/2
11	67.48	0.00556	3.193	3.816	4.363	7/2	0.1948	0.0625	(7/2)*4
12	67.20	0.00232	3.203	3.808	4.348	7/2	0.1940	0.03125	(7/2)*5
13	67.06	0.00071	3.208	3.804	4.348	7/2	0.1939	0.0078125	(7/2)*7
14	67.03	0.00031	3.210	3.804	4.347	7/2	0.1938	0.0039062	(7/2)*8
15	67.00	0.00011	3.210	3.803	4.347	7/2	0.1938	0.0019531	(7/2)*9
16	67.00	1.38-06	3.211	3.803	4.346	7/2	0.0059	0.0002441	(7/2)*12
17*	67.00	1.38-06	3.211	3.803	4.346	7/2	0.0059	--	

\* At the 17<sup>th</sup> iteration,  $\delta_1 = 0.0045$ , and  $\delta_2 = -0.00374$

Table 5. Results of RIA-based RBDO

Iteration No.	Cost	$\beta_1$	$\beta_2$	$\mu_1$	$\mu_2$	No. of Analysis ( $\beta_1/\beta_2$ )	$\ \delta\ $	Step-Size $\alpha$	No. of Analysis for Line Search ( $\beta_1/\beta_2$ )
1	123.	5.549	4.070	5.	6.	45/22	0.2635	1.0	
2	114.81	5.221	4.332	4.850	5.783	22/9	0.2545	1.0	
3	107.17	4.904	4.583	4.704	5.575	26/8	0.2457	1.0	
4	100.00	4.595	4.823	4.562	5.375	15/8	0.2373	1.0	
5	93.41	4.298	5.053	4.423	5.182	15/8	0.2292	1.0	
6	87.21	4.011	5.274	4.288	4.997	9/8	0.2214	1.0	
7	81.43	3.738	5.486	4.157	4.818	20/8	0.2138	1.0	
8	76.04	3.473	5.691	4.030	4.647	21/8	0.2066	1.0	
9	71.00	3.209	5.888	3.906	4.481	12/8	0.1995	0.5	6/8
10	68.63	3.088	5.982	3.845	4.401	9/8	0.0699	1.0	
11	67.00	2.995	6.050	3.802	4.347	4/8	0.0051	1.0	
12	67.08	3.002	6.050	3.801	4.352	4/4	0.0062	1.0	
13*	67.04	3.002	6.056	3.795	4.354	1/4	0.0049	--	

\* At the 13<sup>th</sup> iteration,  $\delta_1 = 0.0025$ ,  $\delta_2 = -.0.00425$

concept of active-in-reliability is implemented here. If a reliability constraint, defined by Eq. (57) satisfies the following condition,

$$G_p^*(\beta_0) \leq \epsilon_G \quad (61)$$

it is called active-in-reliability, because the design point is close to the boundary of the reliability constraint. The value of  $\epsilon_G$  in Eq. (61) is small and positive. A small change in design can move the constraint from feasible to infeasible. The constraint of Eq. (61) can be monitored efficiently with the hybrid method, a PMA, which is the first step of the new RIA. Once the constraint is determined to be active-in-reliability, the new RIA can then be continued to finalize the

constraint value in the form of Eq. (56). With this new implementation, the last example is repeated here and the results are summarized in Table 6. The savings of computation on the second constraint is quite evident. In this example, the value of  $\epsilon_G$  is specified to be 0.5.

Table 6. Results of PMA/RIA-based RBDO

Iteration No.	Cost	$G_{p1}^*$	$G_{p2}$	$\mu_1$	$\mu_2$	No. of Analyses ( $G_{p1}^* / G_{p2}$ )	$\ \delta\ $	Step-Size $\alpha$	No. of Analysis for Line Search ( $G_{p1}^* / G_{p2}$ )
1	123.00	0.971	1.429	5.000	6.000	9/4	0.2635	1.0	
2	114.81	0.779	1.676	4.850	5.783	2/3	0.2545	1.0	
3	107.17	0.618	1.908	4.704	5.575	2/3	0.2457	1.0	
4	100.05	4.596	2.127	2.562	5.375	22/3	0.2373	1.0	
5	93.40	4.298	2.333	4.423	5.182	17/2	0.2292	1.0	
6	87.21	4.016	2.531	4.288	4.997	22/2	0.2214	1.0	
7	81.43	3.750	2.719	4.157	4.818	9/2	0.2138	1.0	
8	76.04	3.467	2.899	4.030	4.647	6/2	0.2066	1.0	
9	71.00	3.210	3.071	3.906	4.481	6/2	0.1995	1.0	
10	68.64	3.087	3.153	3.845	4.402	9/2	0.0699	0.5	9/2
11	67.00	2.996	3.210	3.802	4.347	4/2	0.00501	1.0	
12	67.08	3.002	3.205	3.801	4.352	4/2	0.006109	1.0	
13**	67.04	3.002	3.203	3.796	4.354	1/2	0.004803	--	

\*Starting from the 4<sup>th</sup> iteration, the value is  $\beta_1$  instead of  $G_{p1}$ .

\*\*At the 13<sup>th</sup> iteration,  $\delta_1 = 0.00243$ , and  $\delta_2 = -0.00414$

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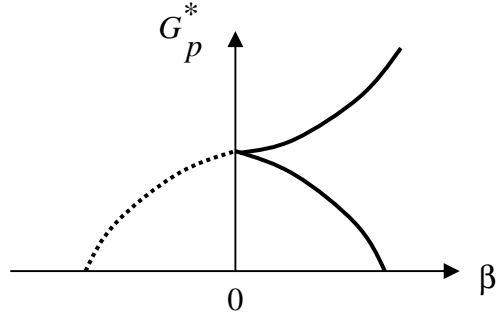
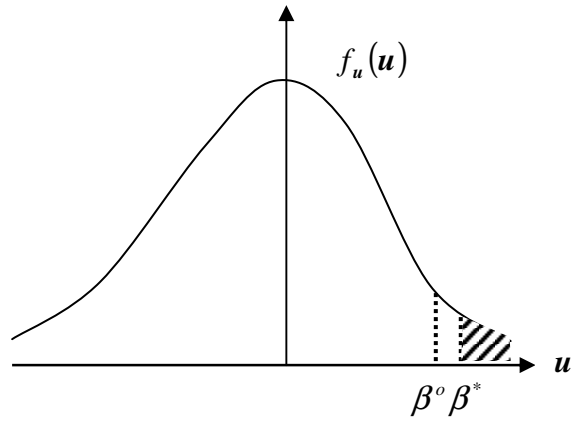
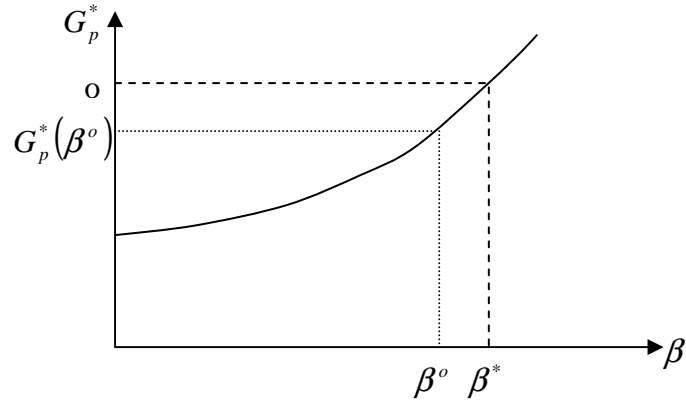


Figure 1 Locus of MPP's in  $G^* \sim \beta$

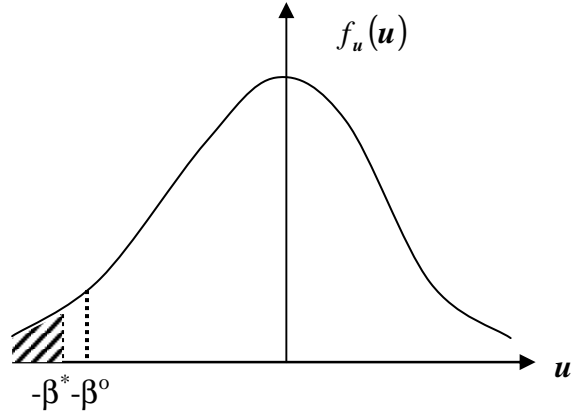


(a) PDF along  $u$  at MPP

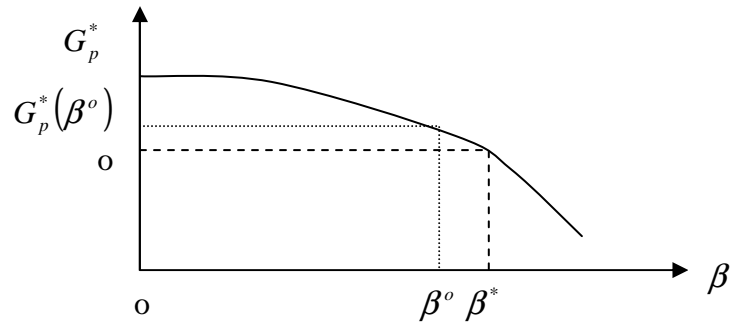


(b) MPP Locus

Figure 2 PDF along  $u$  at MPP and the MPP Locus for  $P_f(G > 0)$



(a) PDF along  $u$  at MPP



(b) MPP Locus

Figure 3 PDF along  $u$  at MPP and the MPP Locus for  $P_f(G < 0)$

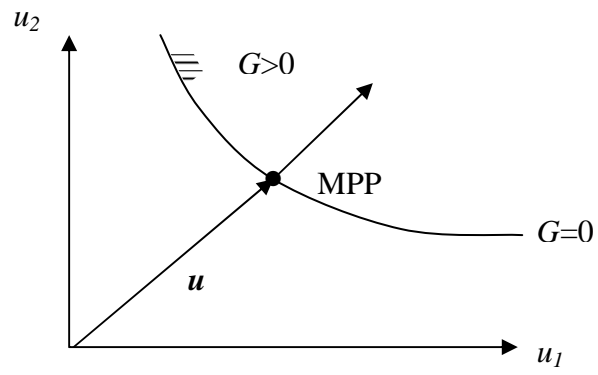


Figure 4 Vectors  $u$  and  $n$  at MPP for  $P_f(G > 0)$

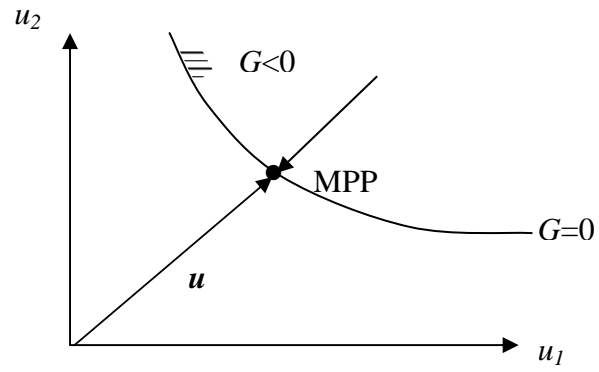


Figure 5 Vectors  $u$  and  $n$  at MPP for  $P_f(G < 0)$



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1. REPORT DATE (DD-MM-YYYY)		2. REPORT TYPE		3. DATES COVERED (From - To)		
01- 03 - 2004		Contractor Report		24-04-2003 to 31-08-2003		
4. TITLE AND SUBTITLE A Most Probable Point-Based Method for Reliability Analysis, Sensitivity Analysis and Design Optimization				5a. CONTRACT NUMBER		
				5b. GRANT NUMBER		
				5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S) Hou, Gene J.-W.				5d. PROJECT NUMBER PO L-18606		
				5e. TASK NUMBER		
				5f. WORK UNIT NUMBER 762-20-61-01		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) NASA Langley Research Center      Old Dominion University Hampton, VA 23681-2199      Department of Mechanical Engineering Norfolk, VA 23529-0247				8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Washington, DC 20546-0001				10. SPONSOR/MONITOR'S ACRONYM(S)  NASA		
				11. SPONSOR/MONITOR'S REPORT NUMBER(S) NASA/CR-2004-213002		
12. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified - Unlimited Subject Category 65 Availability: NASA CASI (301) 621-0390      Distribution: Standard						
13. SUPPLEMENTARY NOTES An electronic version can be found at <a href="http://techreports.larc.nasa.gov/ltrs/">http://techreports.larc.nasa.gov/ltrs/</a> or <a href="http://ntrs.nasa.gov">http://ntrs.nasa.gov</a> Langley Technical Monitor: Perry A. Newman						
14. ABSTRACT  A major step in a most probable point (MPP)-based method for reliability analysis is to determine the MPP. This is usually accomplished by using an optimization search algorithm. The minimum distance associated with the MPP provides a measurement of safety probability, which can be obtained by approximate probability integration methods such as FORM or SORM. The reliability sensitivity equations are derived first in this paper, based on the derivatives of the optimal solution. Examples are provided later to demonstrate the use of these derivatives for better reliability analysis and reliability-based design optimization (RBDO).						
15. SUBJECT TERMS Reliability Analysis, Reliability-Based Design Optimization						
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON	
a. REPORT	b. ABSTRACT	c. THIS PAGE			STI Help Desk (email: <a href="mailto:help@sti.nasa.gov">help@sti.nasa.gov</a> )	
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